PMM U.S.S.R., Vol.52, No.6, pp.809-810, 1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

ON THE USE OF EXTENSIONS OF THE REAL NUMBER FIELD TO SEEK COMPLETELY INTEGRABLE HAMILTONIAN SYSTEMS*

A.A. BUROV

Using methods based on extensions of the real number field, we indicate some classes of completely integrable Hamiltonian systems.

1. Let R_ω be the extension of the field of real numbers whose elements are

$$w = u + \omega v \quad u, v \in R \tag{1.1}$$

where ω is by definition a symbol satisfying the condition $\omega^2 = k$, $k \in R$. The operations of addition, subtraction and multiplication are defined naturally for $w_1, w_2 \in R_\omega$:

$$w_1 \pm w_2 = (u_1 \pm u_2) + \omega (v_1 \pm v_2)$$
$$w_1 \cdot w_2 = (u_1u_2 + kv_1v_2) + \omega (u_1v_2 + u_2v_1)$$

Division is defined for all w_1, w_2 such that $w_2 \notin R_{\omega 0} = \{w : u^2 - kv^2 = 0,\ u, v \in R\}$, in which case

$$\frac{w_1}{w_2} = \frac{u_1u_2 - kv_1v_2}{u_2^2 - kv_2^2} + \omega \frac{u_2v_1 - u_1v_2}{u_2^2 - kv_2^2}$$

If $k = -1$ the set R_ω is simply the complex number field. If $k = 0$ the set R_ω is the object of investigation in screw calculus /1/. If $k < 0$ the set $R_{\omega 0}$ consists of the single number $w = 0 + \omega 0$ while if $k \geqslant 0$ it is the set of divisors of zero in R_ω. We denote by $u = \mathrm{Re}_\omega w$, the real part, and by $\omega v = \omega\, \mathrm{Im}_\omega w$ the ω-imaginary part of a number.

Definition. A function f defined in a domain $G \in R_\omega$ is said to be differentiable at $w_0 \in G$ if for any $\delta > 0$ the limit

$$\frac{\partial f(w_0)}{\partial w} = \lim_{h \to 0} \frac{f(w_0 + h) - f(w_0)}{h}, \quad h \notin \left\{w : \left|\frac{u^2}{v^2} - k\right| < \delta\right\}$$

exists regardless of the way h tends to zero and independently of the parameter δ.

The definition of differentiability is extended in the standard way to the case in which the function is defined in a domain $G \subset R_\omega{}^n = R_\omega \times \ldots \times R_\omega$. In that case, if $w_i = u_i + \omega v_i$, $i = 1, \ldots, n$, then differentiability of a function

$$f(w_1, \ldots, w_n) = \lambda(u_1, \ldots, u_n, v_1, \ldots, v_n) + \omega \nu(u_1, \ldots, u_n, v_1, \ldots, v_n)$$

requires satisfaction of the conditions

$$\partial\lambda/\partial u_i = \partial\nu/\partial v_i, \quad \partial\lambda/\partial v_i = k\partial\nu/\partial u_i \tag{1.2}$$

analogous to the Cauchy-Riemann conditions in the theory of functions of a complex variable.

Consider the system of differential equations

$$\mathbf{w}^{\cdot} = \mathbf{G}(\mathbf{w}) \quad \mathbf{w} \in R_\omega{}^l \tag{1.3}$$

If $\mathbf{w} = \mathbf{u} + \omega\mathbf{v}$, $\mathbf{G}(\mathbf{w}) = \mathbf{g}(\mathbf{u}, \mathbf{v}) + \omega\mathbf{h}(\mathbf{u}, \mathbf{v})$, then system (1.3) may be written

$$\mathbf{u}^{\cdot} = \mathbf{g}(\mathbf{u}, \mathbf{v}),\ \mathbf{v}^{\cdot} = \mathbf{h}(\mathbf{u}, \mathbf{v}) \quad \mathbf{u}, \mathbf{v} \in R^l \tag{1.4}$$

Proposition. Let the differentiable function

$$F(\mathbf{w}) = \varphi(\mathbf{u}, \mathbf{v}) + \omega\, \psi(\mathbf{u}, \mathbf{v}) \tag{1.5}$$

be a first integral of Eqs.(1.3). Then $\varphi(\mathbf{u}, \mathbf{v}), \psi(\mathbf{u}, \mathbf{v})$ are first integrals of Eqs.(1.4).

Proof. Since $F(\mathbf{w})$ is a first integral of Eqs.(1.3),

$$\frac{\partial F}{\partial \mathbf{w}} \cdot \mathbf{G}(\mathbf{w}) = \frac{\partial \varphi}{\partial \mathbf{u}} \mathbf{g}(\mathbf{u}, \mathbf{v}) + k\frac{\partial \psi}{\partial \mathbf{u}} \mathbf{h}(\mathbf{u}, \mathbf{v}) + \omega\left(\frac{\partial \varphi}{\partial \mathbf{u}} \mathbf{h}(\mathbf{u}, \mathbf{v}) + \frac{\partial \psi}{\partial \mathbf{u}} \mathbf{g}(\mathbf{u}, \mathbf{v})\right) \equiv 0$$

Using relations (1.2) and separating the real and ω-imaginary parts, we obtain

$$\frac{\partial \varphi}{\partial \mathbf{u}} \mathbf{g}(\mathbf{u}, \mathbf{v}) + \frac{\partial \varphi}{\partial \mathbf{v}} \mathbf{h}(\mathbf{u}, \mathbf{v}) \equiv 0, \quad \frac{\partial \psi}{\partial \mathbf{v}} \mathbf{h}(\mathbf{u}, \mathbf{v}) + \frac{\partial \psi}{\partial \mathbf{u}} \mathbf{g}(\mathbf{u}, \mathbf{v}) \equiv 0$$

---

*Prikl.Matem.Mekhan., 52, 6, 1036-1038, 1988

Consequently, $\varphi(\mathbf{u}, \mathbf{v})$ and $\psi(\mathbf{u}, \mathbf{v})$ are first integrals of Eqs.(1.5).

2. Consider the system of Hamilton equations

$$\mathbf{Q}^{\cdot} = \partial H/\partial P,\ \mathbf{P}^{\cdot} = -\partial H/\partial \mathbf{Q} \quad \mathbf{P}, \mathbf{Q} \in R_\omega^n \tag{2.1}$$

with differentiable Hamiltonian $H(\mathbf{P}, \mathbf{Q})$.

Proposition. If

$$k \neq 0,\ P_i = p_i + \omega p_{i+n},\ Q_i = q_i + \omega q_{i+n},\ i = 1, \ldots, n$$

$$H(\mathbf{P}, \mathbf{Q}) = \varphi_0(\mathbf{p}, \mathbf{q}) + \omega\psi_0(\mathbf{p}, \mathbf{q})$$

then system (2.1) may be written

$$q_i^{\cdot} = \partial\varphi_0/\partial p_i,\ p_i^{\cdot} = -\partial\varphi_0/\partial q_i \tag{2.2}$$

$$\dot{q}_{i+n} = k^{-1}\partial\varphi_0/\partial p_{i+n},\quad \dot{p}_{i+n} = -k^{-1}\partial\varphi_0/\partial q_{i+n},\quad i = 1, \ldots, n$$

Proof. The function $H(P, Q)$ is differentiable. Consequently, by (1.2),

$$\frac{\partial H}{\partial P_i} = \frac{\partial\varphi_0}{\partial p_i} + \omega k^{-1}\frac{\partial\varphi_0}{\partial p_{i+n}},\quad \frac{\partial H}{\partial Q_i} = \frac{\partial\varphi_0}{\partial q_i} + \omega k^{-1}\frac{\partial\varphi_0}{\partial q_{i+n}} \tag{2.3}$$

Substituting (2.3) into Eqs.(2.1) and separating real and ω-imaginary parts, we obtain the desired assertion.

Eqs.(2.2) form a system of Hamilton equations with $2n$ degrees of freedom, such that when $k = 1$ the symplectic structure turns out to be canonical. Moreover, if system (2.1) has a complete set of differentiable commuting first integrals $J_0 = H, J_1, \ldots, J_{n-1}$, and the functions $\varphi_i = \mathrm{Re}_\omega J_i, \psi_i = \mathrm{Im}_\omega J_i$ are functionally independent then systems (2.2) has a complete set of commuting first integrals and is completely integrable.

3. Consider the system of differentiable equations

$$\mathbf{X}^{\cdot} = \mathbf{X} \times \partial H/\partial \mathbf{X},\ \mathbf{X} \in R_\omega^3 \tag{3.1}$$

with differentiable Hamiltonian $H(\mathbf{X})$. Assume that

$$\mathbf{X} = \boldsymbol{\gamma} + \omega\mathbf{M},\ \boldsymbol{\gamma}, \mathbf{M} \in R^3 \tag{3.2}$$

$$H(\mathbf{X}) = \varphi(\boldsymbol{\gamma}, \mathbf{M}) + \omega\psi(\boldsymbol{\gamma}, \mathbf{M})$$

Since the function (3.2) is differentiable, conditions (1.2) imply

$$\partial H/\partial \mathbf{X} = \partial\varphi/\partial\boldsymbol{\gamma} + \omega\partial\psi/\partial\boldsymbol{\gamma} = \partial\psi/\partial\mathbf{M} + \omega\partial\psi/\partial\boldsymbol{\gamma} \tag{3.3}$$

Substituting (3.3) into Eqs.(3.1) and separating the real and ω-imaginary parts, we obtain

$$\boldsymbol{\gamma}^{\cdot} = \boldsymbol{\gamma} \times \partial\psi/\partial\mathbf{M} + k\mathbf{M} \times \partial\psi/\partial\boldsymbol{\gamma},\quad \mathbf{M}^{\cdot} = \mathbf{M} \times \partial\psi/\partial\mathbf{M} + \boldsymbol{\gamma} \times \partial\psi/\partial\boldsymbol{\gamma} \tag{3.4}$$

When $k = 0$ Eqs.(3.4) form a system of Hamilton equations on the six-dimensional Lie algebra $e(3)$ /2/. It is known that equations of this type describe the motion of various mechanical systems: a solid body with a fixed point in an axially symmetric force field, a solid body in an ideal fluid. Eqs.(3.4) have two trivial first integrals: $J_1 = \mathbf{M}\cdot\boldsymbol{\gamma},\ J_2 = \boldsymbol{\gamma}^2 + k\mathbf{M}^2$. Restriction of the flow (3.4) to their non-singular common level $J_{12}(p, l) = \{J_1 = p, J_2 = l\}$ is described by a system of Hamilton equations with two degrees of freedom. In general, one additional integral is insufficient for a system of type (3.4) to be integrable. In this case, when $\psi = \mathrm{Im}_\omega H$ such an integral indeed exists and it has the form $\varphi = \mathrm{Re}_\omega H$.

The idea of applying extensions of the real field to integrate the equations of motion of mechanical systems goes back to Appell and Lecornu (see /3/). The methods of screw analysis were used to integrate equations of type (3.4) in /4/, where the case of a quadratic function (3.2) is considered.

REFERENCES

1. DIMENTBERG F.M., Screw Calculus and its Applications in Mechanics, Nauka, Moscow, 1965.
2. NOVIKOV S.P., The Hamiltonian formalism and a multivalued analogue of Morse's theory. Uspekhi Mat. Nauk, 27, 5, 1982.
3. APPELL P., Traité de mécanique rationelle, I (2nd Ed.), Gauthier-Villars, Paris, 1902.
4. BUROV A.A. and RUBANOVSKII V.N., On a new solution of equations of the Kirchhoff-Clebsch type. In: Problems in the Investigation of Stability and Stabilization of Motion, Vychisl. Tsentr Akad. Nauk SSSR, Moscow, 1987.

Translated by D.L.